On the reducibility of the Voigt functions

1981 J. Phys. A: Math. Gen. 14 L75

(http://iopscience.iop.org/0305-4470/14/4/002)

J. Phys. A: Math. Gen. **14** (1981) L75–L77. Printed in Great Britain

LETTER TO THE EDITOR

On the reducibility of the Voigt functions

Harold Exton
27 Hollinhurst Avenue, Penwortham, Preston, Lancashire PR1 0AE, UK

Received 7 January 1981

Abstract. The Voigt functions, which are employed in such areas as neutron physics and spectroscopy, are shown to be capable of representation in terms of circular functions and confluent hypergeometric functions.

Recently, Haubold and John (1979) have obtained a number of interesting new results in the analysis of the Voigt functions. These functions are of interest in a number of fields including neutron physics and spectroscopy. Representations due to Reiche (1913) are

$$K(x, y) = \pi^{-1/2} \int_0^\infty \exp(-yr - \tfrac{1}{4}r^2) \cos(xr)\, dr \tag{1}$$

and

$$L(x, y) = \pi^{-1/2} \int_0^\infty \exp(-yr - \tfrac{1}{4}r^2) \sin(xr)\, dr \qquad -\infty < x < \infty \qquad 0 < y. \tag{2}$$

These expressions are used as the starting point of this study. See Haubold and John (1979) for a general review of the Voigt functions.

As is well known, the trigonometric functions of the integrands of (1) and (2) may be written

$$\cos(xr) = {}_0F_1(-; \tfrac{1}{2}; -\tfrac{1}{4}x^2r^2) \tag{3}$$

and

$$\sin(xr) = xr\,{}_0F_1(-; \tfrac{3}{2}; -\tfrac{1}{4}x^2r^2). \tag{4}$$

The generalised hypergeometric function is given by

$${}_pF_q\left(\begin{matrix} a_1, a_2, \ldots, a_p; \\ b_1, b_2, \ldots, b_q; \end{matrix}\, x\right) = \sum_{n=0}^{\infty} \frac{(a_1, n)(a_2, n) \ldots (a_p, n)x^n}{(b_1, n)(b_2, n) \ldots (b_q, n)n!} \tag{5}$$

where

$$(a, n) = a(a+1)(a+2) \ldots (a+n-1) = \Gamma(a+n)/\Gamma(a), \qquad (a, 0) = 1;$$

see Slater (1966, p 40) for example.

In the integrals on the right-hand side of (1) and (2), the trigonometric function and the exponential function e^{-yr} are expanded and the resulting double series integrated term by term. This process is justified on account of the uniform convergence of this series for all finite values of its variables (see Bromwich 1931, p 500).

After a little manipulation, we have the formulae

$$\pi^{1/2}K(x, y) = \sum_{m,n=0}^{\infty} \frac{(-x^2)^m(-2y)^n\Gamma(\frac{1}{2}+m+\frac{1}{2}n)}{(\frac{1}{2}, m)m!n!} \tag{6}$$

and

$$\pi^{1/2}L(x, y) = 2x \sum_{m,n=0}^{\infty} \frac{(-x^2)^m(-2y)^n\Gamma(1+m+\frac{1}{2}n)}{(\frac{3}{2}, m)m!n!} \tag{7}$$

which may be rearranged in terms of the Humbert function Ψ_2 (see Exton 1976, p 28) as follows:

$$\pi^{1/2}K(x, y) = \Gamma(\tfrac{1}{2})\Psi_2(\tfrac{1}{2}; \tfrac{1}{2}, \tfrac{1}{2}; -x^2, y^2) - 2y\Psi_2(1; \tfrac{1}{2}, \tfrac{3}{2}; -x^2, y^2) \tag{8}$$

and

$$\pi^{1/2}(2x)^{-1}L(x, y) = \Psi_2(1; \tfrac{1}{2}, \tfrac{1}{2}; -x^2, y^2) - \Gamma(\tfrac{1}{2})y\Psi_2(\tfrac{3}{2}; \tfrac{3}{2}, \tfrac{3}{2}; -x^2, y^2). \tag{9}$$

The Humbert function Ψ_2 is given by

$$\Psi_2(a; b, b'; x, y) = \sum_{m,n=0}^{\infty} \frac{(a, m+n)x^m y^n}{(b, m)(b', n)m!n!} \tag{10}$$

which is a double hypergeometric function.

One reducible form of this function is (Exton 1976, p 28)

$$\Psi_2(c; c, c; x, y) = e^{x+y}{}_0F_1(-; c; xy) \tag{11}$$

so that the first and second terms respectively of (8) and (9) take the form

$$\exp(y^2 - x^2)\cos(2xy) \tag{12}$$

and

$$\exp(y^2 - x^2)(2xy)^{-1}\sin(2xy) \tag{13}$$

by means of (3) and (4). It is shown below that the remaining Ψ_2 series given above are also reducible to well known functions.

Consider the integral representation

$$\Psi_2(a; b, b'; -x^2, -y^2) = (\Gamma(a))^{-1}\int_0^{\infty} e^{-t}t^{a-1}{}_0F_1(-; b; -x^2t){}_0F_1(-; b'; -y^2t)\,dt \qquad \mathrm{Re}(a) > 0 \tag{14}$$

which may be readily established using term by term integration of the double series expansion of the integrand. If the trigonometric representations of two special ${}_0F_1$ series (3) and (4) are combined with elementary properties of the circular functions, we have

$$\begin{aligned}{}_0F_1(-; \tfrac{1}{2}; -x^2t){}_0F_1(-; \tfrac{1}{2}; -y^2t) &= \cos(2xt^{1/2})\cos(2yt^{1/2}) \\ &= \tfrac{1}{2}[{}_0F_1(-; \tfrac{1}{2}; -t(x-y)^2) + {}_0F_1(-; \tfrac{1}{2}; -t(x+y)^2)]\end{aligned} \tag{15}$$

$$\begin{aligned}{}_0F_1(-; \tfrac{1}{2}; -x^2t){}_0F_1(-; \tfrac{3}{2}; -y^2t) = (2y)^{-1}[&(x-y){}_0F_1(-; \tfrac{3}{2}; -t(x-y)^2) \\ &+ (x+y){}_0F_1(-; \tfrac{3}{2}; -t(x+y)^2)]\end{aligned} \tag{16}$$

and

$$ {}_0F_1(-;\tfrac{3}{2};-x^2t)\,{}_0F_1(-;\tfrac{3}{2};-y^2t)=(8xyt)^{-1}[{}_0F_1(-;\tfrac{1}{2};-t(x-y)^2)-{}_0F_1(-;\tfrac{1}{2};-t(x+y)^2)]. \qquad (17) $$

The application of these results to (14) gives the following reduction formulae:

$$ \Psi_2(a;\tfrac{1}{2},\tfrac{1}{2};-x^2,-y^2)=\tfrac{1}{2}[{}_1F_1(a;\tfrac{1}{2};-(x-y)^2)+{}_1F_1(a;\tfrac{1}{2};-(x+y)^2)] \qquad (18) $$

$$ \Psi_2(a;\tfrac{1}{2},\tfrac{3}{2};-x^2,-y^2)=(2y)^{-1}[(x-y)\,{}_1F_1(a;\tfrac{3}{2};-(x-y)^2)+(x+y)\,{}_1F_1(a;\tfrac{3}{2};-(x+y)^2)] \qquad (19) $$

and, for the sake of completeness,

$$ \Psi_2(a;\tfrac{3}{2},\tfrac{3}{2};-x^2,-y^2)=(8axy)^{-1}[{}_1F_1(a-1;\tfrac{1}{2};-(x-y)^2)-{}_1F_1(a-1;\tfrac{1}{2};-(x+y)^2)]. \qquad (20) $$

Expressions for the Voigt functions which are convenient for computation may easily be deduced from the preceding formulae (8) and (9) together with (12), (13), (18) and (19) as appropriate. These results may be written as

$$ K(x,y)=\exp(y^2-x^2)\cos(2xy) + \pi^{-1/2}[(x-\mathrm{i}y)\,{}_1F_1(1;\tfrac{3}{2};-(x-\mathrm{i}y)^2)+(x+\mathrm{i}y)\,{}_1F_1(1;\tfrac{3}{2};-(x+\mathrm{i}y)^2)] \qquad (21) $$

and

$$ L(x,y)=x\pi^{-1/2}[{}_1F_1(1;\tfrac{1}{2};-(x-\mathrm{i}y)^2)+{}_1F_1(1;\tfrac{1}{2};-(x+\mathrm{i}y)^2)]-\exp(y^2-x^2)\sin(2xy). \qquad (22) $$

The properties of the confluent hypergeometric function ${}_1F_1$ are very well documented; the reader should consult Slater (1960). If desired, the confluent hypergeometric functions in (21) and (22) may be expressed in terms of parabolic cylinder functions.

References

Bromwich T J I'a 1931 *Theory of Infinite Series* 2nd edn (London: Macmillan)
Exton H 1976 *Multiple Hypergeometric Functions and Applications* (Chichester: Ellis Horwood)
Haubold H J and John R W 1979 *Astrophys. Space Sci.* **65** 477
Reiche F 1913 *Ber. Deutsch. Phys. Ges.* **21** 3
Slater L J 1960 *Confluent Hypergeometric Functions* (London: Cambridge University Press)
—— 1966 *Generalised Hypergeometric Functions* (London: Cambridge University Press)